Paper 87-2010

Monty Hall, Paul Erdös, and Monte Carlo

Irvin Snider, Assurant Health, Milwaukee WI

Abstract

You are on a game show. The game show host shows you three doors. Behind one of the doors is a Cadillac. Behind the other two doors are goats. You get to keep the prize behind the door that you pick. You pick a door. The host then opens one of the doors not chosen and reveals a goat. Now the host asks you if you want to stay with your original pick or if you want to switch. What should you do? What difference does it make?

This classic Monty Hall problem confused readers of “Ask Marilyn” and caused intense debate when first published in 1990. Even the famed mathematician Paul Erdös did not get the answer right and was only grudgingly convinced of the correct answer after seeing the output from a Monte Carlo simulation.

This presentation will give a brief history of the Monty Hall problem and will present several methods of deriving the answer, including a SAS® Monte Carlo simulation.

Monty Hall

Monty Hall was the host of the game show *Let’s Make a Deal*. Monty would go into a studio audience filled with contestants who were dressed in outlandish costume and trade with them. The contestants had to weigh offers by Monty as either being a good prize or a “zonk”. For instance, Monty would offer $50 to anyone who had a hard boiled egg. After the exchange of cash for the egg, Monty would then ask the contestant if he would like to trade the money for an unknown but potentially better prize. The contestant could either keep what he had already won or trade it for something that could be a lot better. The catch, of course, is that the trade could result in getting something a lot worse.

The classic Monty Hall problem presents you with three identical doors concealing two goats and one Cadillac. You select a door at random but do not open it. Monty opens a door different from the one that you picked and reveals a goat. At this point of the game Monty always opens a door containing a goat. Monty now gives you the option of sticking with your original selection or switching to the remaining unopened door. You win whatever is behind your final choice. What should you do? Do you switch or do you stick? Does it make a difference?

One would think the odds are 50/50: there are two doors left after Monty opened a door and revealed a goat. That means that one of the two remaining doors has a Cadillac behind it and you have an even chance of winning it. So it does not matter if you switch or if you stick to your original choice. This seems reasonable.

However, it is actually better to switch rather than stick. The odds are two out of three that if you switch your pick, you will win the Cadillac.

Huh? How can that be? This seems counterintuitive.

Ask Marilyn

In 1990 the Monty Hall problem was presented to the readers of Marilyn vos Savant’s column “Ask Marilyn” in *Parade* magazine. Vos Savant, who according to the *Guinness Book of World Records* has the world’s highest IQ, told her readers to switch. She said that sticking with the first choice gives a one-third chance of winning but switching doubles the odds to two-thirds.

Her answer was met with outrage. She was besieged by mail from readers who disagreed with her answer and maintained that the odds were only fifty-fifty. At one point the deluge of mail was nine to one against her. Many of those disagreeing with her answer were professional mathematicians and statisticians. According to some mathematical experts, her answer was indicative of the national crisis in mathematical education. She was accused of contributing to the mathematical illiteracy in this country. She was about to be run out of town by pitch fork wielding mathematicians and torch bearing statisticians.

However, she was correct and a lot of eggheads would soon have egg on their face.

Marilyn’s Three Solutions

The drama of the Monty Hall problem unfolded over several months and several columns in 1990 and 1991. Vos Savant published three solutions to the problem in the attempt to convince her readers that she was correct.

Marilyn Solution Number One

Let's say that instead of three doors there are one million. You pick a door at random since all the doors are identical and there is no indication that any of them differ from any of the others. So you pick door number one. Monty then begins to open doors one at a time, asking you along the way if you want to stick or if you want to switch. You are obstinate and you stick with door number one. Eventually there are two doors left: your original pick and door 777,777.

Doesn't it seem pretty obvious that it is better to switch at this point? Your original pick had a one in a million chance of being correct. You'd have to be pretty darn lucky to pick the winning door at the very beginning of this scenario. To make this argument more convincing, instead of one million doors make it one billion. Still not convinced? Try a trillion.

However, this solution did not convince people and letters of derision poured in. Some were pretty nasty.

Marilyn Solution Number Two

So vos Savant tried again and presented a grid of possible outcomes. In the grid, we assume that you pick door number one. Monty will then open the door containing the goat behind either door two or three. According to the grid, when you switch, you win two times out of three. When you stick, you only win one time out of three.

Choose Door One and Stick with Door One			
Door One	Door Two	Door Three	Outcome
Cadillac	Goat	Goat	Win
Goat	Cadillac	Goat	Lose
Goat	Goat	Cadillac	Lose

Choose Door One and Switch			
Door One	Door Two	Door Three	Outcome
Cadillac	Goat	Goat	Lose
Goat	Cadillac	Goat	Win
Goat	Goat	Cadillac	Win

But the grid of possible outcomes still did not sway the critics.

Marilyn Solution Number Three

Vos Savant tried another approach. Let's say that your original pick is door number one. Now Monty opens a door and reveals a goat. Imagine that just after Monty opened the door and revealed a goat, a UFO lands on the stage and a little green woman emerges. Without knowing what you originally chose, she is asked to pick one of the two unopened doors. The odds of her picking the winning door are fifty-fifty. It is fifty-fifty for her because she lacks the advantage of the original contestant – the help of Monty.

Monty boosts the original contestant's odds by opening a door that contains a goat. Monty always picks the door that contains a goat. If the Cadillac is behind door number two, Monty reveals the goat behind door number three. If the Cadillac is behind door number three, Monty reveals the goat behind door number two. So when you switch, you win the prize if it's behind either door number two or three. YOU WIN EITHER WAY. But if you do not switch, you only win if the prize is behind door number one, your initial pick. So your odds of winning are two out of three if you switch. The little green woman did not have this help when she arrived on stage so her odds of winning are merely fifty-fifty.

Finally, the Ask Marilyn readers were won over with this solution and vos Savant put the controversy behind her. Her critics were forced to eat crow. They were the ones who were "zonked".

Paul Erdös

But not everyone was convinced. One person who needed persuading was Paul Erdös, one of the greatest mathematicians of the 20th century.

Paul Erdös was a poly-mathematician and an eccentric. He traveled the world over regularly appearing unannounced at a fellow mathematician's doorstep, proclaiming that his brain was open. He would then assist with any problem that the interloped was working on. He was one of the most prolific mathematicians who ever lived and collaborated with an amazing number of mathematicians from all over the world. He is considered to be the most published mathematician of all, having published over one thousand papers in many areas of pure mathematics. Like the Kevin Bacon six degrees of separation for Hollywood actors, mathematicians calculate their Erdös number by tracing the degrees of separation through their collaborators back to Paul Erdös -- a degree of one indicating that he or she has published directly with Erdös, and any greater number indicating the number of shared collaborators linking them. The lower the number, the more street credibility given to the mathematician.

Number theorists, who deal with the mathematical properties of whole numbers, consider Erdös to have been one of the world's foremost experts in probability theory.

The story goes that while visiting his friend and fellow mathematician Andrew Vazsonyi, Vazsonyi presented Erdös with the Monty Hall problem during a discussion about the use of probability theory in decision making. When Vazsonyi told Erdös that it was better to switch, he was met with outrage. "No, that is impossible," Erdös said "It should make no difference if you switch."

Uh, Oh. How do you tell one of the greatest mathematicians of the 20th century that he is wrong?

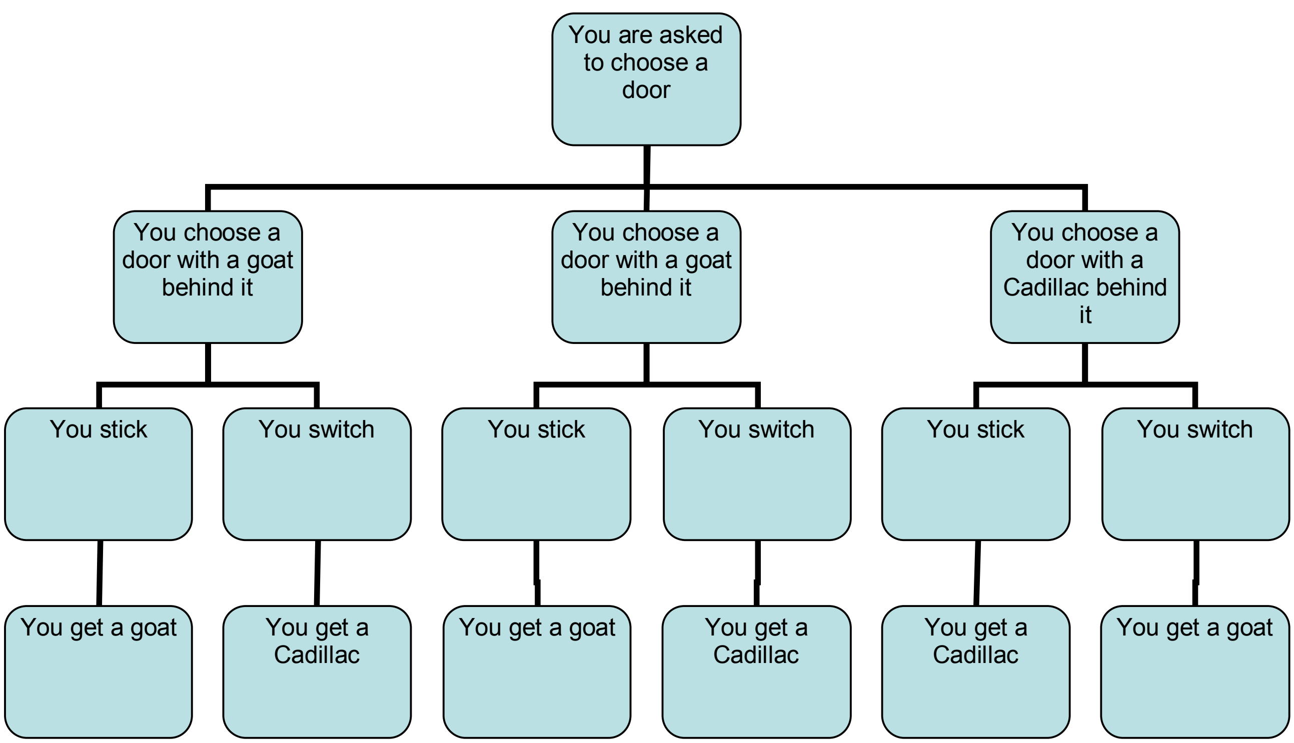
Vazsonyi's Three Solutions

Vazsonyi had several days to convince his doubting houseguest that his assertion was legitimate. Vazsonyi provided three arguments to the problem in the attempt to convince Erdös that he was correct.

Vazsonyi Solution Number One

Vazsonyi's first effort to convince Erdös was with a decision tree. You start by picking a door at the very top of the tree. The second level of the tree indicates the three possible outcomes of your initial decision. At this point Monty gives you the option of sticking or switching. The third level of the tree indicates the choices that you can make which again is switch or stick. The fourth level lists the outcomes of your decisions. As you can hopefully see, if you switched, you win a Cadillac in two out of three cases.

But this decision tree did not convince Erdös.



Vazsonyi Solution Number Two

Vazsonyi tried another approach to convince Erdös that it was better to switch. He reminded Erdös that probability is not a fixed static thing. Probability changes over time. It changes as more information is received. He invoked Bayes' Theorem. Amazingly, according to Vazsonyi, Erdös had no knowledge of Bayes' Theorem. Furthermore, he had no intention of learning about it. Vazsonyi was convinced that Bayes' Theorem was the bridge between pure mathematics and the real world. Erdös was only interested in pure mathematics.

Bayes' Theorem tells us how to properly recalculate the odds after we are given new information. By opening the door that reveals a goat, Monty is disclosing that the Cadillac is not behind that door. This changes the odds.

Bayes' theorem: P(A|B) = (P(B|A)*P(A))/P(B)

P(A) is the prior probability.
P(B) is the marginal probability
P(A|B) is the conditional probability of A, given B
P(B|A) is the conditional probability of B, given A

Initially when the game started, the odds of picking the Cadillac are one out of three. This is known as the prior probability.

Let's say you pick door number one. You have a one out of three chance in winning a Cadillac. The prior probability is 1/3.

Monty then has the choice of opening doors two or three. Remember that Monty knows which doors the goats and the Cadillac are behind. Keep in mind that Monty will never select the door hiding the Cadillac.

If the Cadillac is behind door number one, then the odds of Monty opening door number two are one in two. It does not matter which door he opens since he will reveal a goat in either case.

If the Cadillac is behind door two, then Monty is not going to open it so the odds of him opening door number two are zero.

If the Cadillac is behind door number two, he is certain to open door number three so the odds of him opening door number three are one.

The probability of Monty then opening door number two is:

(1/3 * 1/2) + (1/3 * 0) + (1/3 * 1) = 1/2

This is the marginal probability.

Plug the marginal probability into Bayes' Theorem in order to calculate if it is better to stick:

The probability of the Cadillac being behind door number one given that door number two has been opened is the prior probability times the probability of Monty opening door number two if the prize is behind door number one divided by the marginal probability

or

(1/2 * 1/3) / 1/2 or .333

Plug the marginal probability into Bayes' Theorem in order to calculate if it is better to switch:

The probability of the prize being behind door number three given that Monty has opened door number two is the prior probability times the probability of Monty opening door number two if the prize is behind door number three divided by the marginal probability

or

(1 * 1/3) / 1/2 or .667

As you can see, it is better to switch.

But Erdös would not consider a Bayesian solution. And after struggling with writing this section, I do not blame him.

Vazsonyi Solution Number Three

Having no luck so far, Vazsonyi tried another method. He tried a simulation approach on his computer using the Monte Carlo method. Erdös was more receptive to this method because it was introduced by his good friend Stanislaw Ulam.

The Monte Carlo method involves the use of random sampling techniques and computer simulation to obtain approximate solutions to mathematical problems. Instead of actually watching ten thousand episodes of *Let's Make a Deal* to gather the empirical evidence that switching is always better, Vazsonyi wrote a computer program to simulate the results of one hundred thousand Monty Hall problem outcomes. Monte Carlo simulations are facilitated by the speed of computers.

Vazsonyi ran the program 100,000 times. Erdös watched the results of the simulation. The simulation results indicated that by switching, the odds of winning are indeed two out of three. Finally, he was grudgingly convinced that switching was better. He did not like it but seeing was believing. He could not argue with the results.

Monte Carlo

What follows is a short Monte Carlo simulation written in SAS®.

Monte Carlo Code

```
data _null_;

        switch_the_pick=0;
        stick_with_pick=0;

 do game=1 to 100000;

        seed=7;

        player   =       ceil(ranuni(seed)*3);
        prize    =       ceil(ranuni(seed)*3);

                do until(monty ~= prize and monty ~= player);
                        monty=ceil(ranuni(seed)*3);
                end;

        if monty~=prize and monty~=player then
    do;
      if         player=1 and monty=3 then pick2=2;
      else if    player=1 and monty=2 then pick2=3;
      else if    player=2 and monty=1 then pick2=3;
      else if    player=2 and monty=3 then pick2=1;
      else if    player=3 and monty=1 then pick2=2;
      else if    player=3 and monty=2 then pick2=1;

        if prize   =     pick2    then switch_the_pick      + 1;
        if prize   =     player   then stick_with_pick      + 1;

        if player=1 and monty=3 then
                do;
                if prize=1 then prize1+1;
                if prize=2 then prize2+1;
         end;
    end;
 end;

 switch_winning_pct = switch_the_pick/(stick_with_pick + switch_the_pick);
 stick_winning_pct = stick_with_pick/(stick_with_pick + switch_the_pick);
 numb_obs = stick_with_pick + switch_the_pick;

 put 'Number of Observation = 'numb_obs;
```

```
put;
put 'Wins with initial pick = 'stick_with_pick 'times.';
put;
put 'Wins Switching = ' switch_the_pick 'times.';
put;
put 'Stick winning percent = ' stick_winning_pct;
put;
put 'Switch winning percent = ' switch_winning_pct;
put;

run;
```

Monte Carlo Results

Number of Observation = 100000

Wins with initial pick = 33251 times.

Wins Switching = 66749 times.

Stick winning percent = 0.33251

Switch winning percent = 0.66749

As you can see from 100,000 observations, the switch winning percent is .66.

Conclusion

I like the fact that there are a lot of people that find the Monty Hall problem difficult to understand. I especially like the fact that one of the greatest mathematicians of the 20th century had a hard time getting the right answer. This gives me hope that maybe I have a chance to gain an inkling of understanding in probability.

I do not think that humans are hard wired for thinking in methods of probability. If there were a caveman version of the Monty Hall problem where the caveman had to avoid the tiger behind three rocks, I'm sure that instead of calculating the odds of which rock concealed the tiger, he would simply run away when he heard the first growl. The fight or flee instinct trumps Bayesian probability.

However, I am now convinced that when presented the classic Monty Hall problem, it is better to switch than stick; unless, of course, you are playing the caveman version of the game. Then, as it is with most of my experiences with probability, it is better to run away.

References

Haddon, Mark, The Curious Incident of the Dog in the Night-time, 2003, Random House, Inc., New York, NY

Hoffman, Paul, The Man Who Loved Only Numbers: The Story of Paul Erdös and the Search for Mathematical Truth, 1998, Hyperion, New York, NY

Rosenhouse, Jason, The Monty Hall Problem: The Remarkable Story of Math's Most Contentious Brainteaser, 2009, Oxford University Press, Inc., New York, NY

Vazsonyi, Andrew, Which Door has the Cadillac, 2002, Writers Club Press, Lincoln, NE

Acknowledgements

I do not pretend that the Monte Carlo simulation SAS® code is mine. I thank Kyle Dane for publishing his code in the public domain for all to use and enjoy more than ten years ago.

http://www.listserv.uga.edu/cgi-bin/wa?A2=ind0002a&L=sas-l&F=&S=&P=1184

I thank Emma Snider for her help in editing this paper.

I thank Andrea Everling for her help with the section on Bayes' Theorem.

I thank Dr. Richard O'Farrell for his interest in my paper. Dr. O'Farrell has met Paul Erdös on two occasions. His Erdös number is three.

I thank Deanna Nguyen for her continuing encouragement in another of my attempts to better my communication skills.

Recommended Reading

Fan, Xitao, Felsovalyi, Ákos, Sivo, Stephen A., Keenan, Sean C., SAS® for Monte Carlo Studies, 2001, SAS Institute Inc., Cary, NC

Watts, Duncan J., Six Degrees: The Science of a Connected Age, 2003, W. W. Norton & Company, Inc., New York, NY

For a discussion of Monty Hall and Bayes' Theorem visit:

http://www.maa.org/devlin/devlin_12_05.html

For another example of code that produces a correct answer:

http://sas-and-r.blogspot.com/2010/01/example-723-monty-hall-problem.html

Contact Information

Your comments and questions are valued and encouraged. Contact the author at:

Irvin Snider
Senior Actuarial Analyst
Assurant Health
501 West Michigan Avenue
Milwaukee, WI 53186
Phone: (414) 299-6979
Fax: (414) 299-8043
E-mail: irv.snider@assurant.com
Web: http://www.linkedin.com/in/irvinsnider

Certified Advanced Programmer for SAS®9